Separation of Variables Differential Equations X. Du

- A method of solving certain PDEs. Its applications are limited, mainly because you just have to assume it works to start out. If a contradiction arises, then this method fails to produce a solution.
- Suppose that you can separate your solution into a product of single variable functions.
- We often use separation of variables to reduce a problem to an eigenvalue and eigenfunction problem, from where we can solve the problem as a BVP of an ODE. Then we can apply superposition with Fourier series to obtain the general solution.
- Example: Heat equation in one dimension $u_t = \alpha u_{xx}$.
    - This problem is usually given with boundary conditions. So let's just make this problem homogeneous, $u(0,t) = u(L,t) = 0$, and let's call $u(x,0) = f(x)$.
    - Suppose that your solution takes the form $u(x,t) = X(x)T(t)$
        - Notice that $u(x,t) = X(x)T(t)$ is a product of single variable functions.
    - Substitute. You get $X(x)T'(t) = \alpha X''(x)T(t)$
    - Separate variables: $\frac{X''(x)}{X(x)} = \frac{1}{\alpha}\frac{T'(t)}{T(t)}$.
        - Note that this is only true if both sides are equal to the same constant, i.e. $\frac{X''(x)}{X(x)} = \frac{1}{\alpha}\frac{T'(t)}{T(t)} = -\lambda$.
        - By convention, we denote this constant $-\lambda$. Making this constant negative makes our lives easier (you'll see why very quickly).
    - Now we have two separate ODEs that we can solve.
        - $X''(x) + \lambda X(x) = 0$
        - $T'(t) + \alpha\lambda T(t) = 0$
    - We have now reduced this problem to an eigenfunction problem, where we can just find the eigenfunctions of the first derivative and second derivative operators given whatever boundary conditions we may have.
    - After we find the eigenvalues and eigenfunctions, we must use superposition principle to produce a linear combination to find the general solution that satisfies some $u(x,0) = f(x)$. For that, we use Fourier series. Why? Because it works – we can make a Fourier series converge to $f(x)$.
    - At the end, the solution is $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha}{L^2}t} \sin\frac{n\pi x}{L}$, $c_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$.
- Separation of variables is also used to solve many other linear partial differential equations, such as the wave equation, Laplace equation, and Helmholtz equation.